

# LIPSCHITZ TYPE, RADIAL GROWTH AND DIRICHLET TYPE SPACES ON FUNCTIONS INDUCED BY CERTAIN ELLIPTIC OPERATORS

SHAOLIN CHEN AND ANTTI RASILA

**ABSTRACT.** In this paper, we investigate properties of classes of functions related to certain elliptic operators. Firstly, we prove that a main result of Dyakonov (Acta Math. 178(1997), 143–167) on analytic functions can be extended to this more general setting. Secondly, we study the radial growth on these functions and the obtained results are generalizations of the corresponding results of Makarov (Proc. London Math. Soc. 51(1985), 369–384) and Korenblum (Bull. Amer. Math. Soc. 12(1985), 99–102). Finally, we discuss the Dirichlet type energy integrals on such classes of functions and their applications.

## 1. INTRODUCTION AND MAIN RESULTS

Let  $\mathbb{R}^n$  denote the  $n$ -dimensional Euclidean space, where  $n \geq 2$ . For  $a = (a_1, \dots, a_n)$  and  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , we define the inner product  $\langle \cdot, \cdot \rangle$  by  $\langle x, a \rangle = x_1 a_1 + \dots + x_n a_n$  so that the Euclidean length of  $x$  is defined by  $|x| = \langle x, x \rangle^{1/2} = (|x_1|^2 + \dots + |x_n|^2)^{1/2}$ . Denote a ball in  $\mathbb{R}^n$  with center  $x'$  and radius  $r$  by  $\mathbb{B}^n(x', r) = \{x \in \mathbb{R}^n : |x - x'| < r\}$ . In particular, let  $\mathbb{B}^n = \mathbb{B}^n(0, 1)$  and  $\mathbb{B}_r^n = \mathbb{B}^n(0, r)$ . Set  $\mathbb{D} = \mathbb{B}^2$ , the open unit disk in the complex plane  $\mathbb{C}$ . Let  $\Omega$  be a domain of  $\mathbb{R}^n$ , with non-empty boundary. We use  $d_\Omega(x)$  to denote the Euclidean distance from  $x \in \Omega$  to the boundary  $\partial\Omega$  of  $\Omega$ . If  $\Omega = \mathbb{B}^n$ , we write  $d(x)$  instead of  $d_{\mathbb{B}^n}(x)$ . We denote by  $\mathcal{C}^m(\Omega)$  the set of all  $m$ -times continuously differentiable functions from a domain  $\Omega \subset \mathbb{R}^n$  into  $\mathbb{R}$ , where  $m \in \{1, 2, \dots\}$ . Furthermore, we use  $C$  to denote the various positive constants, whose value may change from one occurrence to another.

Fix  $\tau \geq 1$ , and let  $u \in \mathcal{C}^2(\mathbb{B}^n)$  be a solution to the equation

$$(1.1) \quad \Delta u = \lambda(x)|u|^{\tau-1}u,$$

where  $\Delta$  is the Laplacian operator and  $\lambda$  is a continuous function from  $\mathbb{B}^n$  into  $\mathbb{R}$ . If  $\lambda$  is a constant function in (1.1), then this type equation has attracted the attention of many authors, where the case  $\tau = 1$  and  $\lambda < 0$ , i.e., the *Helmholtz* equation, is particularly important. We refer to [2, 6, 22, 20, 38] and the references therein. If  $\lambda > 0$  is a constant and  $\tau = 1$ , then (1.1) is the *Yukawa equation*, which arose out of an attempt of the Japanese Nobel physicist Hideki Yukawa to describe the nuclear potential of a point charge as  $e^{-\sqrt{\lambda}r}/r$  (cf. [1, 3, 9, 12, 13, 14, 15, 46, 49, 53]). It is

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well known that if  $\lambda$  is a constant function and  $\tau = 1$ , then each solution  $u$  to (1.1) belongs to  $C^\infty(\mathbb{B}^n)$ . Moreover, if  $\lambda = 0$  in (1.1), then  $u$  is harmonic in  $\mathbb{B}^n$ .

In fact, the equation (1.1) can be regarded as the induced equation by the elliptic partial differential operators  $\operatorname{div} p^2 \nabla + q$ , where  $\nabla$  denotes the gradient and  $p, q$  are real-valued functions satisfying  $p \in C^2(\mathbb{B}^n)$  and  $p \neq 0$  in  $\mathbb{B}^n$ . Precisely, the elliptic operators

$$(1.2) \quad E_{p,q} = \operatorname{div} p^2 \nabla + q$$

can be decomposed into the following form (cf. [36, 32])

$$(1.3) \quad E_{p,q} = p(\Delta - \varphi)p,$$

where  $\varphi = (\Delta p)/p - q/p^2$ . By (1.3), we see that the equation

$$(1.4) \quad E_{p,q}(u) = (\operatorname{div} p^2 \nabla + q)u = 0 \text{ in } \mathbb{B}^n$$

is equivalent to *the stationary Schrödinger type equation* (cf. [1, 32])

$$(1.5) \quad \Delta h = \varphi h,$$

where  $h = pu$ . If we can choose some  $p$  and  $q$  such that  $\varphi = \lambda|h|^{\tau-1}$ , then (1.5) is the same type equation as (1.1), where  $\tau \geq 1$ . In particular, if  $n = 2$ , the equation (1.4) is closely related to *the main Vekua equation* (cf. [4, 5, 32, 50])

$$(1.6) \quad \partial_{\bar{z}} w = \frac{\partial_{\bar{z}} f}{f} \bar{w},$$

where  $z = x + iy$ ,  $\partial_z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$  and  $\partial_{\bar{z}} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ . In fact, if  $f = pu_0$ , then, for any solution  $u$  to the equation (1.4), there is a corresponding solution  $w$  to the equation (1.6) such that  $u = \operatorname{Re} w/p$  is a solution to the equation (1.4), where  $u_0$  is a positive solution to the equation (1.4).

**Proposition 1.** *Suppose  $u \in C^2(\mathbb{B}^n)$  is a solution to the equation (1.1), where  $\lambda$  is a nonnegative continuous function from  $\mathbb{B}^n$  into  $\mathbb{R}$  with  $\sup_{x \in \mathbb{B}^n} \lambda(x) < +\infty$ . For all  $x \in \mathbb{B}^n$ , there is a positive constant  $C$  such that*

$$|\nabla u(x)|^\nu \leq \frac{C}{R^{\nu+n}} \left( \int_{\mathbb{B}^n(x,R)} |u(y)|^\nu dy + \int_{\mathbb{B}^n(x,R)} |u(y)|^{\tau\nu} dy \right),$$

where  $\nu \in [1, +\infty)$  and  $R$  is a positive constant such that  $\overline{\mathbb{B}^n(x, R)} \subset \mathbb{B}^n$ .

A continuous increasing function  $\omega : [0, +\infty) \rightarrow [0, +\infty)$  with  $\omega(0) = 0$  is called a *majorant* if  $\omega(t)/t$  is non-increasing for  $t > 0$  (cf. [8, 9, 10, 17, 18, 41]). Given a subset  $\Omega$  of  $\mathbb{R}^n$ , a function  $u : \Omega \rightarrow \mathbb{R}$  is said to belong to the *Lipschitz space*  $L_\omega(\Omega)$  if there is a positive constant  $C$  such that

$$|u(x_1) - u(x_2)| \leq C\omega(|x_1 - x_2|) \quad \text{for all } x_1, x_2 \in \Omega.$$

For  $\nu \in (0, +\infty]$ , the *generalized Hardy space*  $\mathcal{H}_g^\nu(\mathbb{B}^n)$  consists of all those functions  $u : \mathbb{B}^n \rightarrow \mathbb{R}$  such that  $u$  is measurable,  $M_\nu(r, f)$  exists for all  $r \in (0, 1)$  and  $\|u\|_\nu <$

$+\infty$ , where

$$\|u\|_\nu = \begin{cases} \sup_{0 < r < 1} M_\nu(u, r) & \text{if } \nu \in (0, +\infty), \\ \sup_{x \in \mathbb{B}^n} |u(x)| & \text{if } \nu = +\infty, \end{cases} \quad M_\nu(u, r) = \left( \int_{\partial \mathbb{B}^n} |u(r\zeta)|^\nu d\sigma(\zeta) \right)^{\frac{1}{\nu}},$$

and  $d\sigma$  denotes the normalized surface measure on  $\partial \mathbb{B}^n$ .

The classical *harmonic Hardy space*  $\mathcal{H}^p(\mathbb{D})$  consisting of harmonic functions in  $\mathbb{D}$  is a subspace of  $\mathcal{H}_g^p(\mathbb{D})$ .

**Definition 1.** For  $\nu \in (0, +\infty]$ ,  $\alpha > 0$ ,  $\beta \in \mathbb{R}$  and a majorant  $\omega$ , we use  $\mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)$  to denote the *generalized Bloch type space* of all functions  $u \in \mathcal{C}^1(\mathbb{B}^n)$  with  $\|u\|_{\mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)} < +\infty$ , where

$$\|u\|_{\mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)} = \begin{cases} |u(0)| + \sup_{x \in \mathbb{B}^n} \{M_\nu(|\nabla u|, |x|)\omega(\phi(x))\} & \text{if } \nu \in (0, +\infty), \\ |u(0)| + \sup_{x \in \mathbb{B}^n} \{|\nabla u(x)|\omega(\phi(x))\} & \text{if } \nu = +\infty, \end{cases}$$

and  $\phi(x) = d^\alpha(x)(1 - \log d(x))^\beta$ .

It is easy to see that  $\mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)$  is a Banach space for  $\nu \geq 1$ . Moreover, we have the following:

- (1) If  $\beta = 0$ , then  $\mathcal{L}_{+\infty, \omega} \mathcal{B}_\alpha^0(\mathbb{D})$  is called the  $\omega$ - $\alpha$ -Bloch space (cf. [9]).
- (2) If we take  $\alpha = 1$ , then  $\mathcal{L}_{+\infty, \omega} \mathcal{B}_1^\beta(\mathbb{D})$  is called the *logarithmic  $\omega$ -Bloch space*.
- (3) If we take  $\omega(t) = t$  and  $\beta = 0$ , then  $\mathcal{L}_{+\infty, \omega} \mathcal{B}_\alpha^0(\mathbb{D})$  is called the *generalized  $\alpha$ -Bloch space* (cf. [9, 45, 55, 54]).
- (4) If we take  $\omega(t) = t$  and  $\alpha = 1$ , then  $\mathcal{L}_{+\infty, \omega} \mathcal{B}_1^\beta(\mathbb{D})$  is called the *generalized logarithmic Bloch space* (cf. [9, 11, 19, 24, 40, 43, 54]).

In [31], the author studied the Lipschitz spaces on smooth functions. Dyakonov [17] discussed the relationship between the Lipschitz space and the bounded mean oscillation on analytic functions in  $\mathbb{D}$ , and obtained the following result.

**Theorem A.** ([17, Theorem 1]) *Suppose that  $f$  is a analytic function in  $\mathbb{D}$  which is continuous up to the boundary of  $\mathbb{D}$ . If  $\omega$  and  $\omega^2$  are regular majorants, then*

$$f \in L_\omega(\mathbb{D}) \iff (\mathcal{P}_{|f|^2}(z) - |f(z)|^2)^{1/2} \leq C\omega(d(z)),$$

where

$$\mathcal{P}_{|f|^2}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} |f(e^{i\theta})|^2 d\theta,$$

and  $C$  is a positive constant.

In [9, 13], the authors extended Theorem A to complex-valued harmonic functions (see [9, Theorem 4] and [13, Theorem 3]). For the solutions to (1.1), we get the following result, which is a generalization of Theorem A, [9, Theorem 4] and [13, Theorem 3].

**Theorem 1.** *Let  $\alpha \in [1, 2)$  and  $\omega$  be a majorant. Suppose that  $u$  is a solution to (1.1) with  $\tau = 1$ , where  $\lambda$  is a nonnegative constant. Then  $u \in \mathcal{L}_{+\infty, \omega} \mathcal{B}_\alpha^0(\mathbb{B}^n)$  if and only if there is a positive constant  $C$  such that, for all  $r \in (0, d(x)]$ ,*

$$(1.7) \quad \frac{1}{|\mathbb{B}^n(x, r)|} \int_{\mathbb{B}^n(x, r)} |u(y) - u(x)| dy \leq \frac{Cr}{\omega(r^\alpha)},$$

where  $|\mathbb{B}^n(x, r)|$  denotes the volume of  $\mathbb{B}^n(x, r)$ .

Let  $\Omega$  be a proper subdomain of  $\mathbb{R}^n$ . For  $x, y \in \Omega$ , let

$$r_\Omega(x, y) = \frac{|x - y|}{\min\{d_\Omega(x), d_\Omega(y)\}}.$$

The distance ratio metric (see e.g. [51]) is defined for  $x, y \in \Omega$  by setting

$$j_\Omega(x, y) = \log(1 + r_\Omega(x, y)).$$

We say that  $f : \Omega \rightarrow f(\Omega) \subset \mathbb{R}^n$  is *weakly uniformly bounded* in  $\Omega$  (with respect to  $r_\Omega$ ) if there is a constant  $C > 0$  such that  $r_\Omega(x, y) \leq 1/2$  implies  $r_{f(\Omega)}(f(x), f(y)) \leq C$ . For  $x, y \in \Omega$ , let

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{ds}{d_\Omega(x)},$$

where infimum is taken over all rectifiable arcs  $\gamma \subset \Omega$  and  $ds$  stands for the arc length measure on  $\gamma$  (cf. [35, 51]).

In [35], Mateljević and Vuorinen proved the following result.

**Theorem B.** ([35, Theorem 2.8]) *Suppose that  $\Omega$  is a proper subdomain of  $\mathbb{R}^n$  and  $h : \Omega \rightarrow \mathbb{R}^n$  is a harmonic mapping. Then the following conditions are equivalent.*

- (a)  *$h$  is weakly uniformly bounded;*
- (b) *There exists a constant  $C$  such that, for all  $x, y \in G$ ,*

$$k_{u(\Omega)}(u(x), u(y)) \leq Ck_\Omega(x, y).$$

We extended Theorem B to the solutions of (1.1) with  $\tau = 1$ , which is as follows.

**Theorem 2.** *Let  $u = (u_1, \dots, u_n)$  be a vector-valued function from  $\mathbb{B}^n$  into the domain  $u(\mathbb{B}^n) \subset \mathbb{R}^n$  satisfying  $\Delta u_k = \lambda_k u_k$ , where  $k \in \{1, \dots, n\}$  and  $\lambda_k$  is a nonnegative constant. Then the following conditions are equivalent.*

- (1)  *$u$  is weakly uniformly bounded;*
- (2) *There exists a constant  $C$  such that, for all  $x, y \in \mathbb{B}^n$ ,*

$$k_{u(\mathbb{B}^n)}(u(x), u(y)) \leq Ck_{\mathbb{B}^n}(x, y).$$

We remark that we can replace  $\mathbb{B}^n$  by some proper domains  $\Omega \subset \mathbb{R}^n$  in Theorem 2.

Makarov [34] proved that if  $f$  is analytic in  $\mathbb{D}$  with  $\text{Re} f \in \mathcal{L}_{+\infty, \omega} \mathcal{B}_1^0(\mathbb{D})$ , then there is a positive constant  $C$  such that

$$(1.8) \quad \limsup_{r \rightarrow 1^-} \frac{|f(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log \log \log \frac{1}{1-r}}} \leq C \|\text{Re} f\|_{\mathcal{L}_{+\infty, \omega} \mathcal{B}_1^0(\mathbb{D})}$$

for almost every  $\zeta \in \partial\mathbb{D}$ , where  $r \in [0, 1)$  and  $\omega(t) = t$ . In particular, Korenblum [30] showed that if  $u$  is a real harmonic function in  $\mathbb{D}$  with  $u \in \mathcal{L}_{+\infty, \omega} \mathcal{B}_1^0(\mathbb{D})$ , then there is a positive constant  $C$  such that

$$(1.9) \quad \limsup_{r \rightarrow 1^-} \frac{|u(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log \log \frac{1}{1-r}}} \leq C \|u\|_{\mathcal{L}_{+\infty, \omega} \mathcal{B}_1^0(\mathbb{D})}$$

for almost every  $\zeta \in \partial\mathbb{D}$ , where  $\omega(t) = t$ . For related investigations on the radial growth of Bloch type functions, we refer to [7, 21, 25, 26, 27, 44, 48].

Analogously to (1.8) and (1.9), for  $\nu \in (0, +\infty)$  and for functions in  $u \in \mathcal{C}^2(\mathbb{B}^n)$ , satisfying a Bloch-type condition, we prove the following result.

**Theorem 3.** *Let  $\omega$  be a majorant,  $\nu \in [2, +\infty)$ ,  $\alpha > 0$  and  $\beta \leq \alpha$ . Suppose  $u \in \mathcal{C}^2(\mathbb{B}^n)$  satisfying  $u\Delta u \geq 0$  and  $(|\nabla u|^2 + u\Delta u) \in \mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)$ . Then, for  $n \geq 3$  and  $r \in [0, 1)$ ,*

$$M_\nu(u, r) \leq \left[ |u(0)|^2 + \frac{\nu(\nu-1) \| |\nabla u|^2 + u\Delta u \|_{\mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)} r^2}{\omega(1)(n-2)} \int_0^1 \frac{t(1-t^{n-2})}{\phi(tr)} dt \right]^{\frac{1}{2}},$$

where  $\phi$  is defined as in definition 1.

Moreover, for  $n = 2$  and  $r \in [0, 1)$ ,

$$M_\nu(u, r) \leq \left[ |u(0)|^2 + \frac{\nu(\nu-1) \| |\nabla u|^2 + u\Delta u \|_{\mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)} r^2}{\omega(1)} \int_0^1 \frac{t \log \frac{1}{t}}{\phi(tr)} dt \right]^{\frac{1}{2}}.$$

**Definition 2.** For  $m \in \{2, 3, \dots\}$ , we denote by  $\mathcal{HZ}_m(\mathbb{B}^n)$  the class of all functions  $u \in \mathcal{C}^m(\mathbb{B}^n)$  satisfying *Heinz's type nonlinear differential inequality* (cf. [9, 33])

$$(1.10) \quad |\Delta u(x)| \leq a_1(x) |\nabla u(x)|^{b_1} + a_2(x) |u(x)|^{b_2} + a_3(x) \quad \text{for } x \in \mathbb{B}^n,$$

where  $a_k$  ( $k \in \{1, 2, 3\}$ ) are real-valued nonnegative continuous functions in  $\mathbb{B}^n$  and  $b_j$  ( $j \in \{1, 2\}$ ) are nonnegative constants.

**Theorem 4.** *Let  $\omega$  be a majorant,  $\nu \in [2, +\infty)$ ,  $\alpha > 0$ ,  $\beta \leq \alpha$  and  $u \in \mathcal{HZ}_2(\mathbb{B}^n) \cap \mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)$  satisfying  $\sup_{x \in \mathbb{B}^n} a_1(x) < +\infty$ ,  $\sup_{x \in \mathbb{B}^n} a_2(x) < \frac{2n}{\nu}$ ,  $\sup_{x \in \mathbb{B}^n} a_3(x) < +\infty$ ,  $b_1 \in [0, 1]$  and  $b_2 \in [0, 1]$ . If  $n \geq 3$  and  $u\Delta u \geq 0$ , then, for  $r \in [0, 1)$ ,*

$$\begin{aligned} M_\nu(u, r) \leq & \left[ |u(0)|^2 + \frac{\nu(\nu-1)}{(n-2)\omega^2(1)} \|u\|_{\mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)}^2 r^2 \int_0^1 \frac{t(1-t^{n-2})}{\phi^2(rt)} dt \right. \\ & + \frac{\nu \sup_{x \in \mathbb{B}^n} a_1(x)}{(n-2)\omega^{b_1}(1)} \|u\|_{\mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)}^{b_1} r^2 M_\nu(u, r) \int_0^1 \frac{t(1-t^{n-2})}{\phi^{b_1}(rt)} dt \\ & + \frac{\nu \sup_{x \in \mathbb{B}^n} a_2(x)}{2n} r^{2+1+b_2} M_\nu^{1+b_2}(u, r) \\ & \left. + \frac{\nu \sup_{x \in \mathbb{B}^n} a_3(x)}{2n} r^2 M_\nu(u, r) \right]^{\frac{1}{2}}, \end{aligned}$$

where  $\phi$  is defined as in Definition 1.

In particular, if  $n = 2$  and  $u\Delta u \geq 0$ , then, for  $r \in [0, 1)$ ,

$$\begin{aligned} M_\nu(u, r) \leq & \left[ |u(0)|^2 + \frac{\nu(\nu-1)}{\omega^2(1)} \|u\|_{\mathcal{L}_{\nu,\omega}\mathcal{B}_\alpha^\beta(\mathbb{D})}^2 r^2 \int_0^1 \frac{t \log \frac{1}{t}}{\phi^2(rt)} dt \right. \\ & + \frac{\nu \sup_{x \in \mathbb{D}} a_1(x)}{\omega^{b_1}(1)} \|u\|_{\mathcal{L}_{\nu,\omega}\mathcal{B}_\alpha^\beta(\mathbb{D})}^{b_1} r^2 M_\nu(u, r) \int_0^1 \frac{t \log \frac{1}{t}}{\phi^{b_1}(rt)} dt \\ & + \frac{\nu \sup_{x \in \mathbb{D}} a_2(x)}{4} r^2 M_\nu^{1+b_2}(u, r) \\ & \left. + \frac{\nu \sup_{x \in \mathbb{D}} a_3(x)}{4} r^2 M_\nu(u, r) \right]^{\frac{1}{2}}. \end{aligned}$$

We remark that Theorem 4 is a generalization of [9, Theorem 1]. As an application of Theorem 4, we obtain the following result.

**Corollary 1.** *Let  $\omega$  be a majorant,  $\nu \in [2, +\infty)$ ,  $\alpha > 0$  and  $\beta \leq \alpha$ . Suppose that  $u$  is a solution to (1.1) with  $\tau = 1$ , where  $\lambda$  is a nonnegative continuous function from  $\mathbb{B}^n$  into  $\mathbb{R}$  with  $\sup_{x \in \mathbb{B}^n} \lambda(x) < \frac{\nu}{2n}$ . If  $n \geq 3$  and  $u \in \mathcal{L}_{\nu,\omega}\mathcal{B}_\alpha^\beta(\mathbb{B}^n)$ , then, for  $r \in [0, 1)$ ,*

$$M_\nu(u, r) \leq \frac{1}{C^*} \left( |u(0)|^2 + \frac{\nu(\nu-1)}{(n-2)\omega^2(1)} \|u\|_{\mathcal{L}_{\nu,\omega}\mathcal{B}_\alpha^\beta(\mathbb{B}^n)}^2 r^2 \int_0^1 \frac{t(1-t^{n-2})}{\phi^2(rt)} dt \right)^{\frac{1}{2}},$$

where  $C^* = \left(1 - \frac{r^2\nu}{2n} \sup_{x \in \mathbb{B}^n} \lambda(x)\right)^{\frac{1}{2}}$  and  $\phi$  is defined as in definition 1.

In particular, if  $n = 2$  and  $u \in \mathcal{L}_{\nu,\omega}\mathcal{B}_\alpha^\beta(\mathbb{B}^n)$ , then, for  $r \in [0, 1)$ ,

$$M_\nu(u, r) \leq \frac{1}{C^*} \left( |u(0)|^2 + \frac{\nu(\nu-1)}{\omega^2(1)} \|u\|_{\mathcal{L}_{\nu,\omega}\mathcal{B}_\alpha^\beta(\mathbb{D})}^2 r^2 \int_0^1 \frac{t \log \frac{1}{t}}{\phi^2(rt)} dt \right)^{\frac{1}{2}}.$$

For  $\alpha, \gamma, \mu \in \mathbb{R}$ ,

$$\mathcal{D}_{\nabla u}(\alpha, \gamma, \mu) = \int_{\mathbb{B}^n} (1 - |x|^2)^\alpha |\nabla u(x)|^\gamma \left( \sum_{1 \leq j, k \leq n} u_{x_j x_k}^2(x) \right)^\mu dx < +\infty$$

is called a *Dirichlet type energy integral* of  $u$  defined in  $\mathbb{B}^n$  ([9, 12, 13, 22, 28, 47, 48, 52, 53]).

In [13], the authors investigated certain properties on the above Dirichlet type energy integral. In the following, we extend [13, Theorem 4] to a higher order form and give an application.

**Theorem 5.** *Let  $u \in C^2(\mathbb{B}^n)$  be a solution to the equation (1.1) with  $\tau = 1$ . For  $\alpha > 0$ ,  $\mu \in [1, n/2]$  and  $\nu \in [2, +\infty)$ , if  $\mathcal{D}_{\nabla u}(\alpha, 0, \mu) < +\infty$ , then*

$$\int_{\mathbb{B}^n} (d(x))^{\beta\nu} \Delta (|\nabla u(x)|^\nu) dx < +\infty,$$

where  $\beta = \frac{n+\alpha}{2\mu} - 1$ .

We recall that a real function  $f$  is said to have a *harmonic majorant* if there is a positive harmonic function  $F$  in  $\mathbb{B}^n$  such that, for all  $x \in \mathbb{B}^n$ ,  $|f(x)| \leq F(x)$  (cf. [8, 16, 37, 49, 53]). Concerning harmonic majorants, it is well known that a subharmonic function  $u$  defined in  $\mathbb{D}$  has a harmonic majorant if and only if  $\sup_{0 < r < 1} M_1(u, r) < +\infty$  (see [29, Theorem 3.37]). For the solutions to (1.1), we have

**Theorem 6.** *Let  $u \in \mathcal{C}^2(\mathbb{B}^n)$  be a solution to the equation (1.1) with  $\tau = 1$ . Suppose that  $\alpha > 0$ ,  $\mu \in [1, n/2]$  and  $\nu \in [2, +\infty)$  satisfying  $\frac{n+\alpha}{2\mu} - 1 = \frac{1}{\nu}$ . If  $\mathcal{D}_{\nabla u}(\alpha, 0, \mu) < +\infty$ , then  $|\nabla u| \in \mathcal{H}_g^\nu(\mathbb{B}^n)$  and  $|\nabla u|^\nu$  has a majorant.*

The proofs of Proposition 1, and Theorems 1, 2, 3, 4, 5 and 6 will be presented in Section 2.

## 2. PROOFS OF THE MAIN RESULTS

**Lemma 1.** *Let  $u \in \mathcal{C}^2(\mathbb{B}^n)$  with  $u\Delta u \geq 0$  in  $\mathbb{B}^n$ . Then, for  $\nu \geq 1$ ,  $|u|^\nu$  is subharmonic in  $\mathbb{B}^n$ .*

*Proof.* Let  $\mathcal{Z}_u = \{x \in \mathbb{B}^n : u(x) = 0\}$ . Then  $\mathcal{Z}_u$  is a close set, which gives that  $\mathbb{B}^n \setminus \mathcal{Z}_u$  is an open set. By calculations, for  $x \in \mathbb{B}^n \setminus \mathcal{Z}_u$ , we get

$$(2.1) \quad \Delta(|u(x)|^\nu) = \nu(\nu - 1)|u(x)|^{\nu-2}|\nabla u(x)|^2 + \nu|u(x)|^{\nu-2}u(x)\Delta u(x) \geq 0,$$

which implies that  $|u|^\nu$  is subharmonic in  $\mathbb{B}^n$ .  $\square$

**Corollary 2.** *For some  $\tau \geq 1$ , let  $u \in \mathcal{C}^2(\mathbb{B}^n)$  be a solution to the equation (1.1), where  $\lambda$  is a nonnegative continuous function in  $\mathbb{B}^n$ . Then, for  $\nu \geq 1$ ,  $|u|^\nu$  is subharmonic in  $\mathbb{B}^n$ .*

In [42], Pavlović proved the following result.

**Lemma C.** *Suppose that  $\Omega$  is a bounded domain of  $\mathbb{R}^n$  and  $u$  is a subharmonic function in  $\Omega$ . For any  $x \in \Omega$ , let  $r$  be a positive constant such that  $\overline{\mathbb{B}^n(x, r)} \subset \Omega$ . Then, for  $\nu > 0$ , there are positive constant  $C$  such that*

$$|u(x)|^\nu \leq \frac{C}{r^n} \int_{\mathbb{B}^n(x, r)} |u(y)|^\nu dy.$$

The following result is well-known.

**Lemma 2.** *Suppose that  $a, b \in [0, \infty)$  and  $\iota \in (0, \infty)$ . Then*

$$(a + b)^\iota \leq 2^{\max\{\iota-1, 0\}}(a^\iota + b^\iota).$$

**Proof of Proposition 1.** Let  $u \in \mathcal{C}^2(\mathbb{B}^n)$  be a solution to the equation (1.1). Without loss of generality, we assume that  $x = 0$  and  $n \geq 3$ . For  $r \in (0, 1)$  and all  $w \in \mathbb{B}_r^n$ ,

$$(2.2) \quad u(w) = r^{n-2} \left[ \int_{\partial \mathbb{B}^n} P_r(w, \zeta) u(r\zeta) d\sigma(\zeta) - \int_{\mathbb{B}^n} G_r(w, y) \lambda(r y) |u(r y)|^{\tau-1} u(r y) dy \right],$$

where  $V(\mathbb{B}^n)$  is the volume of the unit ball,

$$P_r(w, \zeta) = \frac{r^2 - |w|^2}{|w - r\zeta|^n}$$

is the Poisson kernel and

$$G_r(w, y) = \frac{1}{n(n-2)V(\mathbb{B}^n)} \left[ \frac{1}{|w - ry|^{n-2}} - \frac{1}{(r^2 + |w|^2|y|^2 - 2r \langle w, y \rangle)^{\frac{n-2}{2}}} \right]$$

is the Green function (see [23, 29]). By calculations, we have

$$(2.3) \quad |\nabla P_r(0, \zeta)| = O\left(\frac{1}{r^{n-1}}\right) \text{ and } |\nabla G(0, y)| = O\left(\frac{1}{|ry|^{n-1}}\right).$$

Then, by (2.2) and (2.3), there is a positive constant  $C_1$  such that

$$(2.4) \quad \begin{aligned} |\nabla u(0)| &\leq \frac{C_1}{r} \int_{\partial \mathbb{B}^n} |u(r\zeta)| d\sigma(\zeta) + \frac{C_1}{r} \sup_{\xi \in \mathbb{B}^n} \lambda(\xi) \int_{\mathbb{B}^n} |u(ry)|^{\tau} |y|^{1-n} dy \\ &\leq \frac{C_1 Q_R(|u|)}{r} + \frac{n C_1 V(\mathbb{B}^n) \sup_{\xi \in \mathbb{B}^n} \lambda(\xi) Q_R(|u|^{\tau})}{r}, \end{aligned}$$

which, together with Lemma 2, yield

$$(2.5) \quad |\nabla u(0)|^{\nu} \leq 2^{\nu-1} C_1^{\nu} \left( \frac{Q_R(|u|^{\nu})}{r^{\nu}} + \frac{n^{\nu} \sup_{\xi \in \mathbb{B}^n} \lambda^{\nu}(\xi) Q_R(|u|^{\nu\tau}) V^{\nu}(\mathbb{B}^n)}{r^{\nu}} \right),$$

where  $Q_R(|u|) = \max \{|u(\xi)| : \xi \in \overline{\mathbb{B}_r^n}\}$ .

By (2.5), Corollary 2 and Lemma C, there is a positive constant  $C_2$  such that

$$\begin{aligned} |\nabla u(0)|^{\nu} &\leq 2^{\nu-1} C_1^{\nu} C_2 \left( \frac{1}{r^{\nu+n}} \int_{\mathbb{B}_{2r}^n} |u(y)|^{\nu} dy \right. \\ &\quad \left. + \frac{n^{\nu} V^{\nu}(\mathbb{B}^n) \sup_{\xi \in \mathbb{B}^n} \lambda^{\nu}(\xi)}{r^{n+\nu}} \int_{\mathbb{B}_{2r}^n} |u(y)|^{\tau\nu} dy \right). \end{aligned}$$

The proof of the proposition is complete.  $\square$



**Lemma 3.** For  $\tau = 1$ , let  $u \in \mathcal{C}^2(\mathbb{B}^n)$  be a solution to the equation (1.1), where  $\lambda$  is a nonnegative constant. For all  $a \in \mathbb{B}^n$ , there is a positive constant  $C$  such that

$$|\nabla u(a)| \leq \frac{C}{r} \int_{\partial \mathbb{B}^n} |u(a + r\zeta) - u(a)| d\sigma(\zeta),$$

where  $\overline{\mathbb{B}^n(a, r)} \subset \mathbb{B}^n$ .

*Proof.* For any fixed  $a \in \mathbb{B}^n$ , let  $f(x) = u(x+a) - u(a)$ ,  $x \in \mathbb{B}_r^n$ , where  $r \in [0, d(a))$ . By (2.4) and Corollary 2, there is a positive constant  $C_3$  such that

$$\begin{aligned} |\nabla f(0)| &\leq \frac{C_3}{r} \left( \int_{\partial \mathbb{B}^n} |f(r\zeta)| d\sigma(\zeta) + \int_{\mathbb{B}^n} |f(ry)| |y|^{1-n} dy \right) \\ &= \frac{C_3}{r} \left[ \int_{\partial \mathbb{B}^n} |f(r\zeta)| d\sigma(\zeta) + n \int_0^1 \left( \int_{\partial \mathbb{B}^n} |f(r\rho\zeta)| d\sigma(\zeta) \right) d\rho \right] \\ &= \frac{C_3}{r} \left[ \int_{\partial \mathbb{B}^n} |f(r\zeta)| d\sigma(\zeta) + \frac{n}{r} \int_0^r \left( \int_{\partial \mathbb{B}^n} |f(t\zeta)| d\sigma(\zeta) \right) dt \right] \\ &= \frac{C_3}{r} \left[ \int_{\partial \mathbb{B}^n} |f(r\zeta)| d\sigma(\zeta) + \frac{n}{r} \int_0^r M_1(f, t) dt \right] \\ &\leq \frac{C_3}{r} \left( \int_{\partial \mathbb{B}^n} |f(r\zeta)| d\sigma(\zeta) + n M_1(f, r) \right) \\ &\leq \frac{C_3(1+n)}{r} \int_{\partial \mathbb{B}^n} |f(r\zeta)| d\sigma(\zeta), \end{aligned}$$

which yields

$$|\nabla u(a)| \leq \frac{C}{r} \int_{\partial \mathbb{B}^n} |u(a + r\zeta) - u(a)| d\sigma(\zeta),$$

where  $C = C_3(1+n)$ , completing the proof.  $\square$

**Proof of Theorem 1.** First, we show the “if” part. By Lemma 3, there is a positive constant  $C_4$  such that

$$(2.6) \quad |\nabla u(x)| \leq \frac{C_4}{\rho} \int_{\partial \mathbb{B}^n} |u(x + \rho\zeta) - u(x)| d\sigma(\zeta),$$

where  $\rho \in (0, d(x)]$ . Let  $r = d(x)$ . Multiplying both sides of the inequality (2.6) by  $n\rho^{n-1}$  and integrating from 0 to  $r$ , together with (1.7), we obtain

$$\begin{aligned}
|\nabla u(x)| &\leq \frac{(n+1)C_4}{nr^{n+1}} \int_0^r \left( n\rho^{n-1} \int_{\partial\mathbb{B}^n} |u(x+\rho\zeta) - u(x)| d\sigma(\zeta) \right) d\rho \\
&= \frac{(n+1)C_4}{nr|\mathbb{B}^n(x,r)|} \int_{\mathbb{B}^n(x,r)} |u(y) - u(x)| dy \\
&\leq \frac{(n+1)C_4C}{n} \frac{1}{\omega(r^\alpha)} \\
&= \frac{(n+1)C_4C}{n} \frac{1}{\omega(d^\alpha(x))},
\end{aligned}$$

which implies that  $u \in \mathcal{L}_{+\infty,\omega}\mathcal{B}_\alpha^0(\mathbb{B}^n)$ .

Next we prove the “only if” part. Since  $u \in \mathcal{L}_{+\infty,\omega}\mathcal{B}_\alpha^0(\mathbb{B}^n)$ , we see that, for  $x \in \mathbb{B}^n$ , there is a positive constant  $C_5$  such that

$$(2.7) \quad |\nabla u(x)| \leq \frac{C_5}{\omega(d^\alpha(x))}.$$

For  $x, y \in \mathbb{B}^n$  and  $t \in [0, 1]$ , if  $d(x) > t|x - y|$ , then, by (2.7), we get

$$\begin{aligned}
|u(x) - u(y)| &\leq |x - y| \int_0^1 |\nabla u(x + t(y - x))| dt \\
&\leq C_5 |x - y| \int_0^1 \frac{dt}{\omega(d^\alpha(x + t(y - x)))} \\
&\leq C_5 |x - y| \int_0^1 \frac{dt}{\omega((d(x) - t|x - y|)^\alpha)} \\
&= C_5 \int_0^{|x-y|} \frac{dt}{\omega((d(x) - t)^\alpha)},
\end{aligned}$$

which yields that

$$\begin{aligned}
I &\leq \frac{C_5}{|\mathbb{B}^n(x, r)|} \int_{\mathbb{B}^n(x, r)} \left[ \int_0^{|x-y|} \frac{dt}{\omega((d(x)-t)^\alpha)} \right] dy \\
&= \frac{C_5}{|\mathbb{B}_r^n|} \int_{\mathbb{B}_r^n} \left[ \int_0^{|\xi|} \frac{dt}{\omega((d(x)-t)^\alpha)} \right] d\xi \\
&= \frac{C_5 n}{r^n} \int_0^r \rho^{n-1} \left\{ \int_0^\rho \frac{dt}{\omega((d(x)-t)^\alpha)} \right\} d\rho \\
&\leq \frac{C_5 n}{r^n} \int_0^r \left( \int_t^r \rho^{n-1} d\rho \right) \frac{1}{\omega((r-t)^\alpha)} dt \\
&= \frac{C_5}{r^n} \int_0^r \frac{(r-t)(r^{n-1} + r^{n-2}t + \dots + t^{n-1})}{\omega((r-t)^\alpha)} dt \\
&\leq \frac{C_5 n}{r} \int_0^r \frac{(r-t)}{\omega((r-t)^\alpha)} dt \\
&= \frac{C_5 n}{r} \int_0^r \frac{(r-t)^\alpha}{\omega((r-t)^\alpha)} (r-t)^{1-\alpha} dt \\
&\leq \frac{C_5 n r^{\alpha-1}}{\omega(r^\alpha)} \int_0^r (r-t)^{1-\alpha} dt \\
&= \frac{C_5 n}{(2-\alpha)} \frac{r}{\omega(r^\alpha)},
\end{aligned}$$

where

$$I = \frac{1}{|\mathbb{B}^n(x, r)|} \int_{\mathbb{B}^n(x, r)} |u(y) - u(x)| dy.$$

The proof of this theorem is complete.  $\square$

For an  $n \times n$  real matrix  $A$ , we define the standard *operator norm* by

$$\|A\| = \sup_{x \neq 0} \frac{|Ax|}{|x|} = \max \{ |A\theta| : \theta \in \partial \mathbb{B}^n \}.$$

**Proof of Theorem 2.** We first prove (2)  $\Rightarrow$  (1). Let  $x, y \in \mathbb{B}^n$  with  $r_{\mathbb{B}^n}(x, y) \leq 1/2$ . Then

$$(2.8) \quad |x - y| \leq d(x)/2.$$

By (2.8) and [51, Lemma 3.7], we obtain

$$k_{\mathbb{B}^n}(x, y) \leq 2j_{\mathbb{B}^n}(x, y) \leq 2r_{\mathbb{B}^n}(x, y) \leq 1,$$

which gives that

$$(2.9) \quad k_{u(\mathbb{B}^n)}(u(x), u(y)) \leq C k_{\mathbb{B}^n}(x, y) \leq C.$$

Applying (2.9), we get

$$j_{u(\mathbb{B}^n)}(u(x), u(y)) = \log(1 + r_{u(\mathbb{B}^n)}(u(x), u(y))) \leq k_{u(\mathbb{B}^n)}(u(x), u(y)) \leq C,$$

which implies that  $r_{u(\mathbb{B}^n)}(u(x), u(y)) \leq e^C - 1$ .

Now we prove (1)  $\Rightarrow$  (2). Since  $u$  is weakly uniformly bounded, for every  $x \in \mathbb{B}^n$  and  $y \in \overline{\mathbb{B}^n}(x, d(x)/4)$ , we see that there is a positive constant  $C$ ,

$$(2.10) \quad |u(y) - u(x)| \leq C d_{u(\mathbb{B}^n)}(u(x)).$$

By (2.10) and Lemma 3, we see that there is a positive  $C_6$  such that

$$(2.11) \quad \begin{aligned} \|u'(x)\| &\leq \left( \sum_{k=1}^n |\nabla u_k(x)|^2 \right)^{\frac{1}{2}} \leq \sum_{k=1}^n |\nabla u_k(x)| \\ &\leq \frac{C_6}{r} \int_{\partial \mathbb{B}^n} \sum_{k=1}^n |u_k(x + r\zeta) - u_k(x)| d\sigma(\zeta) \\ &\leq \frac{C_6 \sqrt{n}}{r} \int_{\partial \mathbb{B}^n} |u(x + r\zeta) - u(x)| d\sigma(\zeta) \\ &\leq \frac{C_6 C \sqrt{n}}{r} d_{u(\mathbb{B}^n)}(u(x)), \end{aligned}$$

where  $r = d(x)$  and

$$u'(x) = \begin{pmatrix} \nabla u_1(x) \\ \vdots \\ \nabla u_n(x) \end{pmatrix}.$$

Hence (1)  $\Rightarrow$  (2) follows from (2.11) and [35, Lemma 2.6].  $\square$

**Theorem D.** *Let  $g$  be a function of class  $C^2(\mathbb{B}^n)$ . If  $n \geq 3$ , then for  $r \in (0, 1)$ ,*

$$\int_{\partial \mathbb{B}^n} g(r\zeta) d\sigma(\zeta) = g(0) + \int_{\mathbb{B}^n(0, r)} \Delta g(x) G_n(x, r) dV_N(x),$$

where  $G_n(x, r) = (|x|^{2-n} - r^{2-n})/[n(n-2)]$  and  $dV_N$  is the normalized Lebesgue volume measure in  $\mathbb{B}^n$ . Moreover, if  $n = 2$ , then for  $r \in (0, 1)$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) d\theta = g(0) + \frac{1}{2} \int_{\mathbb{D}_r} \Delta g(z) \log \frac{r}{|z|} dA(z),$$

where  $dA$  denotes the normalized area measure in  $\mathbb{D}$  (cf. [39, 55]).

**Lemma E.** ([9, Lemma 3]) *Suppose that  $\alpha > 0$ ,  $\beta \leq \alpha$  and  $\omega$  is a majorant. Then, for  $r \in (0, 1)$ ,  $\phi(r)$  and  $\phi(r)/\omega(\phi(r))$  are decreasing in  $(0, 1)$ , where  $\phi$  is the same as in Definition 1.*

**Proof of Theorem 3.** Without loss of generality, we assume that  $u$  is a nonzero function and  $n \geq 3$ . By Hölder inequality, for  $\rho \in [0, 1)$ , we have

$$M_{\frac{\nu(\nu-2)}{\nu-1}}^{\frac{\nu(\nu-2)}{\nu-1}}(u, \rho) \leq \left( \int_{\partial \mathbb{B}^n} |u(\rho\zeta)|^\nu d\sigma(\zeta) \right)^{\frac{\nu-2}{\nu-1}} \left( \int_{\partial \mathbb{B}^n} d\sigma(\zeta) \right)^{\frac{1}{\nu-1}},$$

which gives that

$$\begin{aligned} (2.12) \quad & \int_{\partial \mathbb{B}^n} |u(\rho\zeta)|^{\nu-2} (|\nabla u(\rho\zeta)|^2 + u(\rho\zeta)\Delta u(\rho\zeta)) d\sigma(\zeta) \\ & \leq M_{\frac{\nu(\nu-2)}{\nu-1}}^{\nu-2}(u, \rho) \left[ \int_{\partial \mathbb{B}^n} \left( |\nabla u(\rho\zeta)|^2 + u(\rho\zeta)\Delta u(\rho\zeta) \right)^\nu d\sigma(\zeta) \right]^{\frac{1}{\nu}} \\ & \leq M_\nu^{\nu-2}(u, \rho) \left[ \int_{\partial \mathbb{B}^n} \left( |\nabla u(\rho\zeta)|^2 + u(\rho\zeta)\Delta u(\rho\zeta) \right)^\nu d\sigma(\zeta) \right]^{\frac{1}{\nu}}. \end{aligned}$$

By (2.12) and Theorem D, we obtain

$$\begin{aligned} M_\nu^\nu(u, r) &= |u(0)|^\nu + \int_{\mathbb{B}_r^n} \Delta(|u(x)|^\nu) G_n(x, r) dV_N(x) \\ &= |u(0)|^\nu + \int_{\mathbb{B}_r^n} [\nu(\nu-1)|u(x)|^{\nu-2} |\nabla u(x)|^2 \\ &\quad + \nu|u(x)|^{\nu-2} u(x)\Delta u(x)] G_n(x, r) dV_N(x) \\ &\leq |u(0)|^\nu + \nu(\nu-1) \int_0^r \left[ n\rho^{n-1} G_n(\rho, r) \int_{\partial \mathbb{B}^n} |u(\rho\zeta)|^{\nu-2} (|\nabla u(\rho\zeta)|^2 \right. \\ &\quad \left. + u(\rho\zeta)\Delta u(\rho\zeta)) d\sigma(\zeta) \right] d\rho \\ &\leq |u(0)|^\nu + \nu(\nu-1) \int_0^r \left\{ n\rho^{n-1} G_n(\rho, r) M_\nu^{\nu-2}(u, \rho) \right. \\ &\quad \left. \times \left[ \int_{\partial \mathbb{B}^n} \left( |\nabla u(\rho\zeta)|^2 + u(\rho\zeta)\Delta u(\rho\zeta) \right)^\nu d\sigma(\zeta) \right]^{\frac{1}{\nu}} \right\} d\rho \\ &\leq |u(0)|^\nu + \nu(\nu-1) M_\nu^{\nu-2}(u, r) \int_0^r \left\{ n\rho^{n-1} G_n(\rho, r) \right. \\ &\quad \left. \times \left[ \int_{\partial \mathbb{B}^n} \left( |\nabla u(\rho\zeta)|^2 + u(\rho\zeta)\Delta u(\rho\zeta) \right)^\nu d\sigma(\zeta) \right]^{\frac{1}{\nu}} \right\} d\rho, \end{aligned}$$

which, together with the subharmonicity of  $u$  (Corollary 2) and Lemma E, yield that

$$\begin{aligned}
M_\nu^2(u, r) &\leq |u(0)|^2 + \nu(\nu-1) \int_0^r \left\{ n\rho^{n-1} G_n(\rho, r) \right. \\
&\quad \times \left[ \int_{\partial\mathbb{B}^n} \left( |\nabla u(\rho\zeta)|^2 + u(\rho\zeta) \Delta u(\rho\zeta) \right)^\nu d\sigma(\zeta) \right]^{\frac{1}{\nu}} \Big\} d\rho \\
&\leq |u(0)|^2 + \nu(\nu-1) C_7 \int_0^r \frac{n\rho^{n-1} G_n(\rho, r)}{\omega(\phi(\rho))} d\rho \\
&= |u(0)|^2 + \nu(\nu-1) C_7 \int_0^r \frac{n\rho^{n-1} G_n(\rho, r)}{\phi(\rho)} \frac{\phi(\rho)}{\omega(\phi(\rho))} d\rho \\
&\leq |u(0)|^2 + \frac{\nu(\nu-1) C_7}{\omega(1)} \int_0^r \frac{n\rho^{n-1} G_n(\rho, r)}{\phi(\rho)} d\rho \\
&= |u(0)|^2 + \frac{\nu(\nu-1) C_7 r^2}{\omega(1)(n-2)} \int_0^1 \frac{t(1-t^{n-2})}{\phi(tr)} dt,
\end{aligned}$$

where  $C_7 = \| |\nabla u|^2 + u \Delta u \|_{\mathcal{L}_{\nu, \omega} \mathcal{B}_\alpha^\beta(\mathbb{B}^n)}$ . The proof of the theorem is complete.  $\square$

**Proof of Theorem 4.** Without loss of generality, we assume that  $u$  is a nonzero function and  $n \geq 3$ . By Hölder inequality, for  $\rho \in [0, 1)$ , we have

$$(2.13) \quad \int_{\partial\mathbb{B}^n} |u(\rho\zeta)|^{\nu-2} |\nabla u(\rho\zeta)|^2 d\sigma(\zeta) \leq M_\nu^{\nu-2}(u, \rho) M_\nu^2(|\nabla u|, \rho),$$

$$(2.14) \quad \int_{\partial\mathbb{B}^n} |u(\rho\zeta)|^{\nu-1} |\nabla u(\rho\zeta)|^{b_1} d\sigma(\zeta) \leq M_\nu^{\nu-1}(u, \rho) M_{\nu b_1}^{b_1}(|\nabla u|, \rho),$$

$$(2.15) \quad M_{\nu-1+b_2}^{\nu-1+b_2}(u, \rho) \leq \left( \int_{\partial\mathbb{B}^n} |u(\rho\zeta)|^\nu d\sigma(\zeta) \right)^{\frac{\nu+b_2-1}{\nu}} \left( \int_{\partial\mathbb{B}^n} d\sigma(\zeta) \right)^{\frac{1-b_2}{\nu}},$$

$$(2.16) \quad M_{\nu-1}^{\nu-1}(u, \rho) \leq \left( \int_{\partial\mathbb{B}^n} |u(\rho\zeta)|^\nu d\sigma(\zeta) \right)^{\frac{\nu-1}{\nu}} \left( \int_{\partial\mathbb{B}^n} d\sigma(\zeta) \right)^{\frac{1}{\nu}},$$

and

$$(2.17) \quad M_{\nu b_1}^{\nu b_1}(|\nabla u|, \rho) \leq \left( \int_{\partial\mathbb{B}^n} |\nabla u(\rho\zeta)|^\nu d\sigma(\zeta) \right)^{\frac{\nu b_1}{\nu}} \left( \int_{\partial\mathbb{B}^n} d\sigma(\zeta) \right)^{\frac{\nu-\nu b_1}{\nu}}.$$

Applying (2.13), (2.14), (2.15), (2.16), (2.17), [9, Lemma 3] and Theorem D, for  $r \in [0, 1)$ , we get

$$\begin{aligned}
M_\nu^\nu(u, r) &= |u(0)|^\nu + \int_{\mathbb{B}_r^n} \Delta(|u(x)|^\nu) G_n(x, r) dV_N(x) \\
&= \nu(\nu - 1) \int_0^r \left[ n\rho^{n-1} G_n(\rho, r) \int_{\partial\mathbb{B}^n} |u(\rho\zeta)|^{\nu-2} |\nabla u(\rho\zeta)|^2 d\sigma(\zeta) \right] d\rho \\
&\quad + \nu \int_0^r \left[ n\rho^{n-1} G_n(\rho, r) \int_{\partial\mathbb{B}^n} |u(\rho\zeta)|^{\nu-2} u(\rho\zeta) \Delta u(\rho\zeta) d\sigma(\zeta) \right] d\rho \\
&\quad + |u(0)|^\nu \\
&\leq |u(0)|^\nu + \nu(\nu - 1) \int_0^r n\rho^{n-1} G_n(\rho, r) M_\nu^{\nu-2}(u, \rho) M_\nu^2(|\nabla u|, \rho) d\rho \\
&\quad + \nu \sup_{x \in \mathbb{B}^n} a_1(x) \int_0^r n\rho^{n-1} G_n(\rho, r) M_\nu^{\nu-1}(u, \rho) M_{\nu b_1}^{b_1}(|\nabla u|, \rho) d\rho \\
&\quad + \nu \sup_{x \in \mathbb{B}^n} a_2(x) \int_0^r n\rho^{n-1} G_n(\rho, r) M_{\nu-1+b_2}^{\nu-1+b_2}(u, \rho) d\rho \\
&\quad + \nu \sup_{x \in \mathbb{B}^n} a_3(x) \int_0^r n\rho^{n-1} G_n(\rho, r) M_{\nu-1}^{\nu-1}(u, \rho) d\rho \\
&\leq |u(0)|^\nu + \nu(\nu - 1) \int_0^r n\rho^{n-1} G_n(\rho, r) M_\nu^{\nu-2}(u, \rho) M_\nu^2(|\nabla u|, \rho) d\rho \\
&\quad + \nu \sup_{x \in \mathbb{B}^n} a_1(x) \int_0^r n\rho^{n-1} G_n(\rho, r) M_\nu^{\nu-1}(u, \rho) M_{\nu b_1}^{b_1}(|\nabla u|, \rho) d\rho \\
&\quad + \nu \sup_{x \in \mathbb{B}^n} a_2(x) \int_0^r n\rho^{n-1} G_n(\rho, r) M_\nu^{\nu-1+b_2}(u, \rho) d\rho \\
&\quad + \nu \sup_{x \in \mathbb{B}^n} a_3(x) \int_0^r n\rho^{n-1} G_n(\rho, r) M_\nu^{\nu-1}(u, \rho) d\rho,
\end{aligned}$$

which gives that

$$\begin{aligned}
M_\nu^2(u, r) &\leq |u(0)|^2 + \nu(\nu - 1) \int_0^r n\rho^{n-1} G_n(\rho, r) M_\nu^2(|\nabla u|, \rho) d\rho \\
&\quad + \nu M_\nu(u, r) \sup_{x \in \mathbb{B}^n} a_1(x) \int_0^r n\rho^{n-1} G_n(\rho, r) M_{\nu b_1}^{b_1}(|\nabla u|, \rho) d\rho \\
&\quad + \nu M_\nu^{1+b_2}(u, r) \sup_{x \in \mathbb{B}^n} a_2(x) \int_0^r n\rho^{n-1} G_n(\rho, r) d\rho \\
&\quad + \nu M_\nu(u, r) \sup_{x \in \mathbb{B}^n} a_3(x) \int_0^r n\rho^{n-1} G_n(\rho, r) d\rho
\end{aligned}$$

$$\begin{aligned}
&= |u(0)|^2 + \frac{\nu(\nu-1)r^2}{n-2} \int_0^1 t(1-t^{n-2}) M_\nu^2(|\nabla u|, rt) dt \\
&\quad + \frac{\nu r^2 \sup_{x \in \mathbb{B}^n} a_1(x)}{n-2} M_\nu(u, r) \int_0^1 t(1-t^{n-2}) M_\nu^{b_1}(|\nabla u|, rt) dt \\
&\quad + \frac{\nu r^2 \sup_{x \in \mathbb{B}^n} a_2(x)}{2n} M_\nu^{1+b_2}(u, r) \\
&\quad + \frac{\nu r^2 \sup_{x \in \mathbb{B}^n} a_3(x)}{2n} M_\nu(u, r) \\
&\leq |u(0)|^2 + \frac{\nu(\nu-1)}{n-2} \|u\|_{\mathcal{L}_{\nu, \omega}^\beta(\mathbb{B}^n)}^2 r^2 \int_0^1 \frac{\phi^2(rt)}{\omega^2(\phi(rt))} \frac{t(1-t^{n-2})}{\phi^2(rt)} dt \\
&\quad + \frac{\nu \sup_{x \in \mathbb{B}^n} a_1(x)}{n-2} \|u\|_{\mathcal{L}_{\nu, \omega}^\beta(\mathbb{B}^n)}^{b_1} r^2 M_\nu(u, r) \int_0^1 \frac{\phi^{b_1}(rt)}{\omega^{b_1}(\phi(rt))} \frac{t(1-t^{n-2})}{\phi^{b_1}(rt)} dt \\
&\quad + \frac{\nu \sup_{x \in \mathbb{B}^n} a_2(x)}{2n} r^2 M_\nu^{1+b_2}(u, r) \\
&\quad + \frac{\nu \sup_{x \in \mathbb{B}^n} a_3(x)}{2n} r^2 M_\nu(u, r) \\
&\leq |u(0)|^2 + \frac{\nu(\nu-1)}{(n-2)\omega^2(1)} \|u\|_{\mathcal{L}_{\nu, \omega}^\beta(\mathbb{B}^n)}^2 r^2 \int_0^1 \frac{t(1-t^{n-2})}{\phi^2(rt)} dt \\
&\quad + \frac{\nu \sup_{x \in \mathbb{B}^n} a_1(x)}{(n-2)\omega^{b_1}(1)} \|u\|_{\mathcal{L}_{\nu, \omega}^\beta(\mathbb{B}^n)}^{b_1} r^2 M_\nu(u, r) \int_0^1 \frac{t(1-t^{n-2})}{\phi^{b_1}(rt)} dt \\
&\quad + \frac{\nu \sup_{x \in \mathbb{B}^n} a_2(x)}{2n} r^2 M_\nu^{1+b_2}(u, r) \\
&\quad + \frac{\nu \sup_{x \in \mathbb{B}^n} a_3(x)}{2n} r^2 M_\nu(u, r).
\end{aligned}$$

The proof of the theorem is complete.  $\square$

**Lemma 4.** *Let  $u \in \mathcal{C}^2(\mathbb{B}^n)$  be a solution to the equation (1.1) with  $\tau = 1$ . Then, for  $\nu \geq 1$ ,  $\left(\sum_{1 \leq k, j \leq n} u_{x_k x_j}^2\right)^\nu$  is subharmonic in  $\mathbb{B}^n$ .*

*Proof.* Without loss of generality, we assume that  $U = \left(\sum_{1 \leq k, j \leq n} u_{x_k x_j}^2\right)^\nu$  has no zeros. By computations, we get

$$\begin{aligned}
\Delta U &= 4\nu(\nu-1) \left(\sum_{1 \leq k, j \leq n} u_{x_k x_j}^2\right)^{\nu-2} \left(\sum_{1 \leq m, k, j \leq n} u_{x_k x_j} u_{x_k x_j x_m}\right)^2 \\
&\quad + 2\nu \left(\sum_{1 \leq k, j \leq n} u_{x_k x_j}^2\right)^{\nu-1} \sum_{1 \leq k, j \leq n} \left[(\Delta u)_{x_k x_j} u_{x_k x_j} + \sum_{m=1}^n u_{x_m x_k x_j}^2\right]
\end{aligned}$$



$$\begin{aligned}
&= 4\nu(\nu-1) \left( \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2 \right)^{\nu-2} \left( \sum_{1 \leq m, k, j \leq n} u_{x_k x_j} u_{x_k x_j x_m} \right)^2 \\
&\quad + 2\nu \left( \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2 \right)^{\nu-1} \sum_{1 \leq k, j \leq n} \left[ \lambda u_{x_k x_j}^2 + \sum_{m=1}^n u_{x_m x_k x_j}^2 \right] \geq 0,
\end{aligned}$$

which implies that  $U$  is subharmonic in  $\mathbb{B}^n$ .  $\square$

**Proof of Theorem 5.** By Lemma C and Lemma 4, for  $\mu \in [1, n/2]$  and  $x \in \mathbb{B}^n$ , there is a positive constant  $C_8$  such that

$$\left( \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2(x) \right)^\mu \leq \frac{C_8}{(d(x))^{n+\alpha}} \int_{\mathbb{B}^n(x, \frac{d(x)}{2})} (1-|y|)^\alpha \left( \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2(y) \right)^\mu dy,$$

which gives that

$$(2.18) \quad \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2(x) \leq \frac{(C_8 \mathcal{D}_{\nabla u}(\alpha, 0, \mu))^{\frac{1}{\mu}}}{(d(x))^{\frac{n+\alpha}{\mu}}}.$$

Let

$$H_u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \frac{\partial^2 u}{\partial x_2^2} & \cdots & \frac{\partial^2 u}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \frac{\partial^2 u}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}$$

be the Hessian matrix of  $u$ . Then

$$(2.19) \quad \|H_u\| \leq \sqrt{\sum_{1 \leq k, j \leq n} u_{x_k x_j}^2}.$$

By (2.18) and (2.19), we get

$$\begin{aligned}
(2.20) \quad |\nabla u(x)| &\leq |\nabla u(0)| + \int_{[0, x]} \|H_u(y)\| |dy| \\
&\leq |\nabla u(0)| + \int_{[0, x]} \frac{(C_8 \mathcal{D}_{\nabla u}(\alpha, 0, \mu))^{\frac{1}{2\mu}}}{(d(y))^{\frac{n+\alpha}{2\mu}}} |dy| \\
&\leq |\nabla u(0)| + \frac{C_9}{(d(x))^{\frac{n+\alpha}{2\mu}-1}},
\end{aligned}$$

where

$$C_9 = \frac{2\mu(C_8\mathcal{D}_{\nabla u}(\alpha, 0, \mu))^{\frac{1}{2\mu}}}{n + \alpha - 2\mu},$$

and  $[0, x]$  is the line segment from 0 to  $x$ .

Applying (2.20) and Lemma 2, for  $\nu \geq 2$ , we have

$$(2.21) \quad \begin{aligned} |\nabla u(x)|^{\nu-2} &\leq \left[ |\nabla u(0)| + \frac{C_9}{(d(x))^\beta} \right]^{\nu-2} \\ &\leq 2^{\nu-2} \left[ |\nabla u(0)|^{\nu-2} + \frac{C_9^{\nu-2}}{(d(x))^{\beta(\nu-2)}} \right] \end{aligned}$$

and

$$(2.22) \quad \begin{aligned} |\nabla u(x)|^\nu &\leq \left[ |\nabla u(0)| + \frac{C_9}{(d(x))^\beta} \right]^\nu \\ &\leq 2^\nu \left[ |\nabla u(0)|^\nu + \frac{C_9^\nu}{(d(x))^{\beta\nu}} \right], \end{aligned}$$

where  $\beta = \frac{n+\alpha}{2\mu} - 1$ .

We divide the remaining part of the proof into two cases, namely  $\nu \in [2, 4)$  and  $\nu \in [4, +\infty)$ .

**Case I :** Let  $\nu \in [4, +\infty)$ . By direct computations, we see that

$$(2.23) \quad \begin{aligned} \Delta(|\nabla u|^\nu) &= \nu(\nu-2)|\nabla u|^{\nu-4} \sum_{j=1}^n \left( \sum_{k=1}^n u_{x_k x_j} u_{x_k} \right)^2 \\ &\quad + \nu |\nabla u|^{\nu-2} \sum_{k=1}^n u_{x_k} (\Delta u)_{x_k} + \nu |\nabla u|^{\nu-2} \sum_{j=1}^n \sum_{k=1}^n u_{x_k x_j}^2 \\ &\leq \nu(\nu-2)|\nabla u|^{\nu-2} \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2 + \lambda \nu |\nabla u|^\nu \\ &\quad + \nu |\nabla u|^{\nu-2} \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2 \\ &= \nu(\nu-1)|\nabla u|^{\nu-2} \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2 + \lambda \nu |\nabla u|^\nu. \end{aligned}$$

It follows from (2.21), (2.22) and (2.23) that

$$\begin{aligned}
(2.24) \quad (d(x))^{\beta\nu} \Delta(|\nabla u|^\nu) &\leq \nu(\nu-1)(d(x))^{\beta\nu} |\nabla u|^{\nu-2} \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2 \\
&\quad + \lambda\nu (d(x))^{\beta\nu} |\nabla u|^\nu \\
&= \nu(\nu-1)(d(x))^{\beta\nu - \frac{\alpha}{\mu}} |\nabla u|^{\nu-2} (d(x))^{\frac{\alpha}{\mu}} \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2 \\
&\quad + \lambda\nu (d(x))^{\beta\nu} |\nabla u|^\nu \\
&\leq C_{10} (d(x))^{\frac{\alpha}{\mu}} \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2 + C_{11},
\end{aligned}$$

where  $C_{10} = 2^{\nu-2}\nu(\nu-1)(|\nabla u(0)|^{\nu-2} + C_9^{\nu-2})$  and  $C_{11} = 2^\nu \lambda\nu (|\nabla u(0)|^\nu + C_9^\nu)$ .

By Hölder's inequality, we obtain

$$\begin{aligned}
(2.25) \quad \int_{\mathbb{B}^n} (d(x))^{\frac{\alpha}{\mu}} \left( \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2(x) \right) dx &\leq (\mathcal{D}_{\nabla u}(\alpha, 0, \mu))^{\frac{1}{\mu}} \left( \int_{\mathbb{B}^n} dx \right)^{1-\frac{1}{\mu}} \\
&= (V(\mathbb{B}^n))^{1-\frac{1}{\mu}} (\mathcal{D}_{\nabla u}(\alpha, 0, \mu))^{\frac{1}{\mu}}.
\end{aligned}$$

By (2.24) and (2.25), we conclude that

$$\begin{aligned}
(2.26) \quad \int_{\mathbb{B}^n} (d(x))^{\beta\nu} \Delta(|\nabla u(x)|^\nu) dx &\leq \int_{\mathbb{B}^n} \left[ C_{10} (d(x))^{\frac{\alpha}{\mu}} \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2(x) + C_{11} \right] dx \\
&\leq C_{10} (V(\mathbb{B}^n))^{2-\frac{1}{\mu}} (\mathcal{D}_{\nabla u}(\alpha, 0, \mu))^{\frac{1}{\mu}} + C_{11} V(\mathbb{B}^n) \\
&< +\infty.
\end{aligned}$$

**Case II :** Let  $\nu \in [2, 4)$ . In this case, for  $m \in \{1, 2, \dots\}$ , we let  $f_m^\nu = (|\nabla u|^2 + \frac{1}{m})^{\frac{\nu}{2}}$ . It is not difficult to know that  $\Delta(f_m^\nu)$  is integrable in  $\mathbb{B}_r^n$ . Then, by (2.24), (2.26) and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
\lim_{m \rightarrow +\infty} \int_{\mathbb{B}^n} (d(x))^{\beta\nu} \Delta(f_m^\nu(x)) dx &= \int_{\mathbb{B}^n} (d(x))^{\beta\nu} \lim_{m \rightarrow +\infty} \Delta(f_m^\nu(x)) dx \\
&\leq \int_{\mathbb{B}^n} \left[ C_{10} (d(x))^{\frac{\alpha}{\mu}} \sum_{1 \leq k, j \leq n} u_{x_k x_j}^2(x) + C_{11} \right] dx \\
&\leq C_{10} (V(\mathbb{B}^n))^{2-\frac{1}{\mu}} (\mathcal{D}_{\nabla u}(\alpha, 0, \mu))^{\frac{1}{\mu}} + C_{11} V(\mathbb{B}^n) \\
&< +\infty.
\end{aligned}$$

The proof of the theorem is complete.  $\square$

**Lemma 5.** Let  $u \in \mathcal{C}^3(\mathbb{B}^n)$  with  $\sum_{k=1}^n u_{x_k}(\Delta u)_{x_k} \geq 0$  in  $\mathbb{B}^n$ . Then, for  $\nu \geq 1$ ,  $|\nabla u|^\nu$  is subharmonic in  $\mathbb{B}^n$ .

*Proof.* Let  $\mathcal{Z}_{\nabla u} = \{x \in \mathbb{B}^n : |\nabla u(x)| = 0\}$ . Then  $\mathbb{B}^n \setminus \mathcal{Z}_{\nabla u}$  is an open set. For  $j \in \{1, \dots, n\}$  and  $x \in \mathbb{B}^n \setminus \mathcal{Z}_{\nabla u}$ , we have

$$\begin{aligned} (|\nabla u(x)|^\nu)_{x_j x_j} &= \nu(\nu - 2)|\nabla u(x)|^{\nu-4} \left( \sum_{k=1}^n u_{x_k x_j}(x) u_{x_k}(x) \right)^2 \\ &\quad + \nu |\nabla u(x)|^{\nu-2} \sum_{k=1}^n \left( u_{x_k x_j x_j}(x) u_{x_k}(x) + u_{x_k x_j}^2(x) \right), \end{aligned}$$

which gives that

$$\begin{aligned} \Delta(|\nabla u(x)|^\nu) &= \nu(\nu - 2)|\nabla u(x)|^{\nu-4} \sum_{j=1}^n \left( \sum_{k=1}^n u_{x_k x_j}(x) u_{x_k}(x) \right)^2 \\ &\quad + \nu |\nabla u(x)|^{\nu-2} \sum_{k=1}^n u_{x_k}(x) (\Delta u(x))_{x_k} + \nu |\nabla u(x)|^{\nu-2} \sum_{j=1}^n \sum_{k=1}^n u_{x_k x_j}^2(x) \\ &\geq 0. \end{aligned}$$

Therefore, for  $\nu \geq 1$ ,  $|\nabla u|^\nu$  is subharmonic in  $\mathbb{B}^n$ .  $\square$

The following result easily follows from Lemma 5.

**Corollary 3.** *Let  $u \in \mathcal{C}^3(\mathbb{B}^n)$  be a solution to the equation (1.1), where  $\lambda$  is a nonnegative constant. Then, for  $\nu \geq 1$ ,  $|\nabla u|^\nu$  is subharmonic in  $\mathbb{B}^n$ .*

**Proof of Theorem 6.**  $|\nabla u| \in \mathcal{H}_g^\nu(\mathbb{B}^n)$  follows from [12, Theorem 1] and Theorem 5.

Next we prove  $|\nabla u|^\nu$  has a harmonic majorant. For  $x \in \mathbb{B}^n$ , let

$$G_r(x) = \int_{\partial \mathbb{B}^n} \frac{1 - |x|^2}{|x - \zeta|^n} |\nabla u(r\zeta)|^\nu d\sigma(\zeta),$$

where  $r \in [0, 1)$ . By Corollary 3, we see that  $|\nabla u|^\nu$  is subharmonic in  $\mathbb{B}^n$ , which, together with  $|\nabla u| \in \mathcal{H}_g^\nu(\mathbb{B}^n)$ , imply that

$$|\nabla u(rx)|^\nu \leq \int_{\partial \mathbb{B}^n} \frac{1 - |x|^2}{|x - \zeta|^n} |\nabla u(r\zeta)|^\nu d\sigma(\zeta) = G_r(x) < +\infty$$

and  $G_r(0) = M_\nu^\nu(|\nabla u|, r) < +\infty$ . For  $x \in \mathbb{B}^n$ , applying the Harnack Theorem to the sequence  $\{G_{1-1/m}(x)\}_{m=1}^\infty$ , we see that

$$G(x) = \lim_{m \rightarrow +\infty} G_{1-1/m}(x)$$

is also a harmonic function in  $\mathbb{B}^n$ . Hence  $|\nabla u|^\nu$  has a harmonic majorant in  $\mathbb{B}^n$ . The proof of the theorem is complete.  $\square$

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S. CHEN, COLLEGE OF MATHEMATICS AND STATISTICS, HENGYANG NORMAL UNIVERSITY,  
HENGYANG, HUNAN 421008, PEOPLE'S REPUBLIC OF CHINA.

*E-mail address:* `mathechen@126.com`

A. RASILA, DEPARTMENT OF MATHEMATICS AND SYSTEMS ANALYSIS, AALTO UNIVERSITY,  
P. O. Box 11100, FI-00076 AALTO, FINLAND.

*E-mail address:* `antti.rasila@iki.fi`